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# k-NETWORKS, AND COVERING PROPERTIES OF CW-COMPLEXES

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We assume that all spaces are  $T_2$ . First of all, we shall recall some definitions.

Let  $X$  be a space, and let  $\mathcal{C}$  be a cover of  $X$ . Then  $X$  is determined by  $\mathcal{C}$  [3] (or  $X$  has the weak topology with respect to  $\mathcal{C}$  in the usual sense), if  $F \subset X$  is closed in  $X$  if and only if  $F \cap C$  is closed in  $C$  for every  $C \in \mathcal{C}$ . Here, we can replace "closed" by "open". Every space is determined by an open cover.  $X$  is dominated by  $\mathcal{C}$ , if the union of any subcollection  $\mathcal{C}'$  of  $\mathcal{C}$  is closed in  $X$ , and the union is determined by  $\mathcal{C}'$ .

Let  $X$  be a space, and  $\mathcal{P}$  be a cover of  $X$ . Then  $\mathcal{P}$  is a k-network, if whenever  $K \subset U$  with  $K$  compact and  $U$  open in  $X$ , then  $K \subset \bigcup \mathcal{P}' \subset U$  for some finite  $\mathcal{P}' \subset \mathcal{P}$ . If we replace "compact" by "single point" then such a cover is called "net (or network)". k-networks have played a role in  $\mathcal{K}_0$ -spaces (i.e., regular spaces with a countable k-network) and  $\mathcal{K}$ -spaces (i.e., regular spaces with a  $\sigma$ -locally finite k-network).

Let  $\mathcal{A} = \{A_\alpha; \alpha \in A\}$  be a collection of subsets of a space  $X$ . Then  $\mathcal{A}$  is closure-preserving if  $\overline{\bigcup \{A_\alpha; \alpha \in B\}} = \bigcup \{\overline{A_\alpha}; \alpha \in B\}$  for any  $B \subset A$ .  $\mathcal{A}$  is hereditarily closure-preserving if  $\overline{\bigcup \{B_\alpha; \alpha \in B\}} = \bigcup \{\overline{B_\alpha}; \alpha \in B\}$  whenever  $B \subset A$  and  $B_\alpha \subset A_\alpha$  for each  $\alpha \in B$ . Every space is dominated by a hereditarily closure-preserving closed cover.

A  $\sigma$ -hereditarily closure-preserving collection is the union of countably many hereditarily closure-preserving collections. We shall use " $\sigma$ -CP (resp.  $\sigma$ -HCP)" instead of " $\sigma$ -closure-preserving (resp.  $\sigma$ -hereditarily closure-preserving)".

$\mathcal{A}$  is point-finite (resp. point-countable) if every  $x \in X$  is in at most finitely (resp. countably) many element of  $\mathcal{A}$ .

The concept of CW-complexes due to J. H. Whitehead [5] is well-known. A space  $X$  is a CW-complex if it is a complex with cells  $\{e_\lambda; \lambda\}$  satisfying (a) and (b) below.

(a) Each cell  $e_\lambda$  is contained in a finite subcomplex of  $X$ .

(b)  $X$  is determined by the closed cover  $\{\bar{e}_\lambda; \lambda\}$  of  $X$ .

We note that every  $\bar{e}_\lambda$  is not a subcomplex.

As is well-known, every CW-complex  $X$  is dominated by the cover of all finite subcomplexes of  $X$ , hence  $X$  is dominated by a cover of compact metric subsets of  $X$ .

Let  $\{e_\lambda; \lambda\}$  be the cells of a CW-complex  $X$ . We shall say that  $\{e_\lambda; \lambda\}$  is  $(\sigma-)$  locally finite;  $(\sigma-)$  HCP, etc., if so is respectively the collection  $\{e_\lambda; \lambda\}$  of subsets of  $X$ . We note that the collection  $\{e_\lambda; \lambda\}$  is  $(\sigma-)$  locally finite;  $(\sigma-)$  CP;  $(\sigma-)$  HCP if and only if so is respectively  $\{\bar{e}_\lambda; \lambda\}$ .

Results. Let  $X$  be a CW-complex with cells  $\{e_\lambda; \lambda\}$ . Then the following hold. (a) is well-known, and (b) is due to [2].

(a)  $X$  is a paracompact, and  $\sigma$ -space (i.e.,  $X$  has a  $\sigma$ -locally finite net).

(b)  $X$  is an  $M_1$ -space (in the sense of [2]), hence  $X$  has a  $\sigma$ -CP  $k$ -network.

(c)  $X$  has a point-countable  $k$ -network.

However, every CW-complex is not a metric space (not even a Fréchet space, nor an  $\aleph_1$ -space). We have the following characterizations of  $X$ . Recall that a space is Fréchet, if whenever  $x \in \overline{A}$ , there exists a sequence in  $A$  converging to the point  $x$ . (A) is well-known, and (B) is due to [4].

(A)  $X$  is a metric space if and only if  $\{e_\lambda; \lambda\}$  is locally finite.

(B)  $X$  is a Fréchet space if and only if  $\{e_\lambda; \lambda\}$  is HCP.

(C)  $X$  is an  $\aleph_1$ -space if and only if  $\{e_\lambda; \lambda\}$  is  $\sigma$ -locally finite.

(D)  $X$  has a  $\sigma$ -HCP  $k$ -network if and only if  $\{e_\lambda; \lambda\}$  is  $\sigma$ -HCP.

(E)  $X$  is a symmetric space (in the sense of [1]) if and only if  $\{\overline{e}_\lambda; \lambda\}$  is point-finite.

(F)  $X$  has a point-countable closed  $k$ -network if and only if  $\{\overline{e}_\lambda; \lambda\}$  is point-countable.

Remark. Let  $X$  be a CW-complex with cells  $\{e_\lambda; \lambda\}$ .

(1) The property " $\{\bar{e}_\lambda; \lambda\}$  is HCP" need not imply that  $X$  has a point-countable closed  $k$ -network, and not imply that  $\{\bar{e}_\lambda; \lambda\}$  is point-countable.

(2) The property " $\{e_\lambda; \lambda\}$  is CP" need not imply that  $X$  has a CP or  $\sigma$ -HCP  $k$ -network, and not imply that  $\{e_\lambda; \lambda\}$  is  $\sigma$ -HCP.

(3) The property " $X$  is a symmetric space with a  $\sigma$ -CP  $k$ -network" need not imply that  $X$  has a  $\sigma$ -HCP  $k$ -network, and not imply that  $\{e_\lambda; \lambda\}$  is  $\sigma$ -CP.

Question. Let  $X$  be a CW-complex with cells  $\{e_\lambda; \lambda\}$ .

Characterize " $\{e_\lambda; \lambda\}$  is CP (or  $\sigma$ -CP)" by means of a nice topological property of  $X$ .

Finally, concerning spaces dominated by compact metric subsets, similarly to CW-complexes the following analogue holds.

Let  $X$  be a space dominated by a cover  $\{X_\lambda; \lambda\}$  with each  $E_\lambda$  compact metric. Here,  $E_0 = X_0$ ,  $E_\lambda = X_\lambda - \bigcup \{X_\mu; \mu < \lambda\}$ . Then it is possible to replace  $\{e_\lambda; \lambda\}$  (or  $\{\bar{e}_\lambda; \lambda\}$ ) by  $\{E_\lambda; \lambda\}$  (or  $\{\bar{E}_\lambda; \lambda\}$ ) in (A)  $\sim$  (F).

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